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A CNOIDAL WAVE IN A PLANE WALL JET OF AN INCOMPRESSIBLE VISCOUS FLUID†

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It is shown that the situation of a non-classical boundary layer with a self-induced pressure is realized in the sublayer of a tangential jet stream adjacent to a plane solid surface, where a zone of perturbed non-linear motion is localized. In fact, the existence of a class of comparatively large amplitude perturbations, when a flow of this type acquires a multistage structure, has been established. The assumptions under which, in the case of finite pulsation amplitudes, the evolution of the wave fields obeys the Korteweg–de Vries equation are discussed. A non-linear oscillating solution of the Korteweg–de Vries equation is considered in the form of a cnoidal wave, which provides an example of a periodic critical layer adjacent to the wall past which the flow occurs. Under the assumptions are made, the above-mentioned critical layer decomposes into a main non-linear inviscid part and a thin viscous boundary sublayer. The following result is formulated: the condition for the existence of a periodic solution in the viscous sublayer reduces the set of permissible values of the cnoidal wave parameters.

1. INTRODUCTION

A steady two-dimensional flow in a laminar jet of an incompressible viscous fluid which is bounded from below by a plane screen has a tangential component of the velocity $u^* = u^*_\infty U_0(\tilde{x}, Y_m)$ with the profile show in Fig. 1. Henceforth, dimensional quantities are indicated with asterisks and the maximum velocity in some fixed section of the jet $x^* = L^*_\infty$, for example, can be chosen as the constant u^*_∞ defining the velocity scale. The domains $x^* = L^*_\infty(1 + \tilde{x})$ will be considered, the length of which $L^*_\infty\tilde{x}$, $\tilde{x} \ll 1$ are much smaller than the characteristics length L^*_∞ of the flow along the x^* axis, coinciding with the solid wall, of the Cartesian system of coordinates $\{x^*, y^*\}$. The function U_0 is therefore subsequently considered to depend solely on the single spatial coordinate along the normal to the wall Y_m.

In order to clarify what has been said above, we will introduce the Reynolds number $\mathrm{Re} = \rho^*_\infty u^*_\infty L^*_\infty/\mu^*_\infty$, where ρ^*_∞ and μ^*_∞ are the density and coefficient of viscosity of the fluid and we will assume that $\mathrm{Re} \to \infty$. Then, the assumption that a steady flow is formed by viscous forces implies an estimate $\mathrm{Re}^{-1/2}L^*_\infty$ for the characteristic dimension of the jet along the transverse coordinate y^* (as a corollary of the balancing of the inertial and viscous terms in the Navier–Stokes equations). This estimate means that the (fast) variable $Y_m = \mathrm{Re}^{1/2}L^{*-1}_\infty y^*$ is of the order of unity in the main part of the jet. The derivatives of the function U_0 with respect to the spatial coordinates $\tilde{x} = (x^* - L^*_\infty)L^{*-1}_\infty$, $\tilde{y} = y^*L^{*-1}_\infty$ are of different orders:

$$\frac{\partial U_0}{\partial \tilde{x}} = O(1), \quad \frac{\partial U_0}{\partial \tilde{y}} = \mathrm{Re}^{1/2}\frac{\partial U_0}{\partial Y_m} = O(\mathrm{Re}^{1/2})$$

When account is taken of the last formulae for the derivatives, the equation of continuity yields the quantity $\upsilon^* u^{*-1}_\infty = O(\mathrm{Re}^{-1/2})$ for the vertical component υ^* of the velocity of the steady motion of the fluid in the jet.

The estimate for υ^* given above, and also for the derivatives of the velocity U_0 illustrates the fact that the streamlines in the jet are almost parallel to the plane wall and, consequently, a dependence of the function U_0 on the longitudinal variable $\tilde{x}$ is only manifested at distances which are comparable

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with the length scale L_∞^*, and is unimportant in domains with a length of the order of the thickness of the jet $\mathrm{Re}^{-1/2}L_\infty^*$, $\mathrm{Re} \to \infty$. In this sense, the jet stream being considered is analogous to the flow in a boundary layer with the sole difference that the velocity not only vanishes on the wall but also in the outer layer of the jet:

$$U_0 \to 0, \quad Y_m \to 0; \quad U_0 = \lambda_1 Y_m + \frac{1}{2}\lambda_2 Y_m^2 + \dots, \quad Y_m \to 0 \tag{1.1}$$

On introducing, in addition to the above-mentioned variables $\tilde{x}$, $\tilde{y}$, the time $\tilde{t} = L_\infty^{*-1}u_\infty^* t^*$, the components $\tilde{u} = u^* u_\infty^{*-1}$, $\tilde{\upsilon} = \upsilon^* u_\infty^{*-1}$ of the velocity vector $\tilde{\mathbf{v}} = \{\tilde{u} = \tilde{\upsilon}\}$ and the excess pressure $\tilde{p} = (p^* - p_\infty^*)\rho_\infty^{*-1}u_\infty^{*-2}$, where p_∞^* is the pressure on the outer edge of the jet, we write the Navier–Stokes equations in the dimensionless form

$$\frac{\partial\tilde{\mathbf{v}}}{\partial\tilde{t}} + (\tilde{\mathbf{v}}\cdot\nabla)\tilde{\mathbf{v}} = -\nabla\tilde{p} + \mathrm{Re}^{-1}\nabla^2\tilde{\mathbf{v}}, \quad (\nabla\cdot\tilde{\mathbf{v}}) = 0, \quad \nabla = \left\{\frac{\partial}{\partial\tilde{x}}, \frac{\partial}{\partial\tilde{y}}\right\} \tag{1.2}$$

We shall seek a solution of Eqs (1.2) when $\mathrm{Re} \to \infty$ as the sum of a main steady solution of the form of (1.1) and a perturbation which has been introduced into the flow by some means or other. Suppose $\lambda\!\!\!^-$ is the characteristics length (along the $\tilde{x}$ axis) of the perturbation wave and that the amplitude of the perturbation of the longitudinal component of the velocity $\tilde{u}$ is of the order of $\alpha \ll 1$.

Since the main part of the jet has a thickness of the order of $\mathrm{Re}^{-1/2}$, the estimate for the vertical component of the velocity

$$\tilde{\upsilon} = O(\mathrm{Re}^{-1/2}\alpha\lambda\!\!\!^{-\,-1}) \tag{1.3}$$

follows from the equation of continuity and, from the projection of the equation of motion (1.2) onto the $\tilde{y}$ axis, taking account of this estimate, we obtain an estimate for the pressure perturbation

$$\tilde{p} = O(\mathrm{Re}^{-1}\alpha\lambda\!\!\!^{-\,-2}) \tag{1.4}$$

In the non-linear domain close to the surface past which the flow occurs, where, by virtue of the condition when $Y_m \to 0$ in relations (1.1), the main velocity U_0 is of the order of its own perturbation, projection of the equation of motion onto the $\tilde{x}$ axis gives

$$\tilde{p} = O(\alpha^2) \tag{1.5}$$

Comparing estimates (1.4) and (1.5), we obtain an estimate for the length of the perturbation wave

$$\lambda\!\!\!^- = O(\mathrm{Re}^{-1/2}\alpha^{-1/2}) \tag{1.6}$$

while the magnitude of the characteristic time

$$\tilde{t} = O(\mathrm{Re}^{-1/2}\alpha^{-3/2}) \tag{1.7}$$

follows from the equations of motion.

If the independent parameters Re and α satisfy the condition

$$\alpha\mathrm{Re} \gg 1$$

than, by virtue of estimate (1.6), which determines the longitudinal scale of the perturbations, the characteristic length is

$$\lambda\!\!\!^- \ll 1 \tag{1.8}$$

Condition (1.8) will be satisfied everywhere below which, as already pointed out, enables us to ignore the dependence of the function U_0 on the variable $\tilde{x}$ in domains with a length of the order of $\lambda\!\!\!^-$. In accordance with this, we put $U_0 = U_0(Y_m)$ and use estimates (1.3)–(1.8), which fix the class of perturbations being considered, as guiding considerations for the expansion of the Navier–Stokes equations in formal asymptotic series.

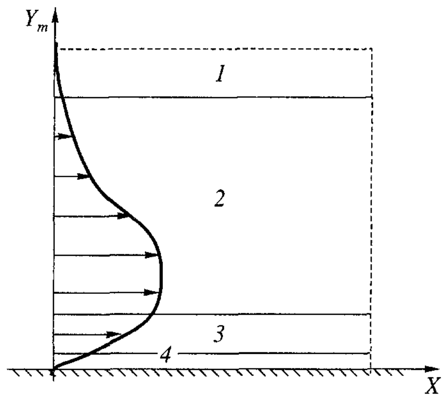

Fig. 1

The above asymptotic estimates of the space–time and amplitude parameters of the perturbations are characteristic of the so-called free interaction region [1–5]. The concept of free interaction was developed [6–9] for wall jets and, also, for flows with similar profiles for the longitudinal component of the velocity. Here, the asymptotic description was based on the separation within the jet of two layers, arranged one above the other (levels or decks) and, also, a weakly perturbed domain which is immediately adjacent to the jet from above (domain *1* in Fig. 1). A generalization of this triple level-model to unsteady motions has been proposed in [10]. A triple level structure does not arise for arbitrary $\alpha \to 0$ and $\mathrm{Re} \to \infty$, but for value which are associated by the completely determinate order relation [6–10]

$$\alpha = \mathrm{Re}^{-1/7} \tag{1.9}$$

Below we consider the case when the amplitude parameter α, while remaining small, is greater than the order of magnitude of the estimate from the right-hand side of relation (1.9). An increase in the amplitude parameter leads to a change in the structure of the interaction region, namely the separation of a lower non-linear boundary sublayer (in the above-mentioned triple level model) into viscous and non-viscous parts, and the pattern of the flow field therefore becomes a quadruple-level pattern. It will be seen from what follows that, in this case, the mechanism of the evolution of the perturbations is described by inviscid equations which are integrated independently of the equations for the viscous boundary sublayer. However, the existence of a regular solution (for all finite points) in the viscous sublayer actually serves as an internal criterion for realizing this perturbed flow scheme which has been modified compared with the scheme available in the literature [6–10].

2. THE QUADRUPLE-LEVEL ASYMPTOTIC STRUCTURE OF THE PERTURBED VELOCITY FIELD

In accordance with estimates (1.6) and (1.7), we introduce the new independent variables

$$T = \mathrm{Re}^{1/2}\alpha^{3/2}\tilde{t}, \quad X = \mathrm{Re}^{1/2}\alpha^{1/2}\tilde{x}, \quad Y_m = \mathrm{Re}^{1/2}\tilde{y} \tag{2.1}$$

and, in the main layer of the stream, that is, in domain *2* in Fig. 1, where $[T, X, Y_m] = O(1)$, we represent the solution of the Navier–Stokes equations in the form

$$\begin{aligned} \tilde{u} &= U_0(Y_m) + \alpha u_{1m}(T, X, Y_m) + \alpha^2 u_{2m}(T, X, Y_m) + \ldots \\ \tilde{v} &= \alpha^{3/2} v_{1m}(T, X, Y_m) + \alpha^{5/2} v_{2m}(T, X, Y_m) + \ldots \\ \tilde{p} &= \alpha^2 p_{1m}(T, X, Y_m) + \alpha^3 p_{2m}(T, X, Y_m) + \ldots \end{aligned} \tag{2.2}$$

We now introduce expansion (2.2) into Eqs (1.2) and retain the leading terms in these equations. Then, we have

$$U_0\frac{\partial u_{1m}}{\partial X}+v_{1m}\frac{dU_0}{dY_m}=0,\quad U_0\frac{\partial v_{1m}}{\partial X}=-\frac{\partial p_{1m}}{\partial Y_m},\quad \frac{\partial u_{1m}}{\partial X}+\frac{\partial v_{1m}}{\partial Y_m}=0 \tag{2.3}$$

as the system of equations of the first approximation.

The solution of system (2.3) is determined, apart from the arbitrary function $A_1=A_1(T,X)$, in the following manner

$$u_{1m}=A_1(T,X)\frac{dU_0}{dY_m},\quad v_{1m}=-\frac{\partial A_1}{\partial X}U_0(Y_m),\quad p_{1m}=\frac{\partial^2 A_1}{\partial X^2}\int\limits_{+\infty}^{Y_m}U_0^2(\xi)d\xi \tag{2.4}$$

Expressions (2.4) are obtained taking account of the condition $p_{1m}\to 0$ on the outer boundary ($Y_m\to+\infty$) of domain *2*. The arbitrary function must satisfy the condition $A_1(T,X)\to 0$ when $X\to-\infty$, which ensures that the perturbations $[u_{1m},v_{1m},p_{1m}]\to 0$ decay at infinity upstream along the flow from the domain being considered. The meaning of solution (2.4) lies in the fact that the angular coefficient $\partial\tilde{y}(\tilde{t},\tilde{x})/\partial\tilde{x}=\tilde{v}\tilde{u}^{-1}$ of a streamline $\tilde{y}=\tilde{y}(\tilde{t},\tilde{x})$ of the perturbed jet stream in domain *2* is, according to expansions (2.2), equal to $O(\alpha^{5/2})$ with an accuracy $\alpha^{3/2}v_{1m}U_0^{-1}=-\alpha^{3/2}\partial A_1/\partial X$. Integration of the relation

$$\frac{\partial\tilde{y}(\tilde{t},\tilde{x})}{\partial\tilde{x}}=\alpha^{1/2}\frac{\partial Y_m(T,X)}{\partial X}=\frac{\tilde{v}}{\tilde{u}}$$

gives the equation of the streamline

$$Y_m=Y_{-\infty}-\alpha A_1(T,X)$$

The constant of integration $Y_{-\infty}$ fixes the streamline, giving its position when $X\to-\infty$, $A_1(T,X)\to 0$. In fact, the parameter $Y_{-\infty}$ establishes the asymptotic form of the streamline in the unperturbed flow when $X\to-\infty$, where the velocity vector is collinear with the wall: $\tilde{u}=U_0(Y_m)$, $\tilde{v}=0$. Hence, the function $A_1(T,X)$ is the magnitude of the instantaneous displacement of a streamline in domain *2* with respect to its unperturbed position.

Using the second relation of (1.1), we write the limiting form of solution (2.2), (2.4) when $Y_m\to 0$, that is near the restricting flow of the solid surface

$$\tilde{u}=\lambda_1Y_m+\lambda_1\alpha A_1+\ldots,\quad \tilde{v}=-\alpha^{3/2}\lambda_1\frac{\partial A_1}{\partial X}Y_m+\ldots,\quad \tilde{p}=-\alpha^2\frac{\partial^2 A_1}{\partial X}\int\limits_0^{+\infty}U_0^2(\xi)d\xi+\ldots \tag{2.5}$$

The first two terms of the asymptotic representation (2.5) for $\tilde{u}$ become of the same order of smallness if $Y_m=O(\alpha)$. The last estimate gives the thickness of the non-linear sublayer located in the bottom part of domain *2*, where the perturbation of the velocity is of the same order as the velocity itself. We shall call this sublayer domain *3* (Fig. 1). The scales for the time and longitudinal coordinate are the same in domains *2* and *3* and the new transverse coordinate

$$Y_a=\alpha^{-1}Y_m \tag{2.6}$$

in domain *3* is of the order of unity. Formulae (2.5) show that, instead of representation (2.2), it is necessary to seek a solution in the non-linear domain *3* in the form

$$\tilde{u}=\alpha u_{1a}(T,X,Y_a)+\ldots,\quad \tilde{v}=\alpha^{5/2}v_{1a}(T,X,Y_a)+\ldots,\quad \tilde{p}=\alpha^2p_{1a}(T,X,Y_a)+\ldots \tag{2.7}$$

Substituting expressions (2.7) into the system of Navier–Stokes equations (1.2), we find

$$\frac{\partial u_{1a}}{\partial T}+u_{1a}\frac{\partial u_{1a}}{\partial X}+v_{1a}\frac{\partial u_{1a}}{\partial Y_a}=-\frac{\partial p_{1a}}{\partial X}+\alpha^{-7/2}\mathrm{Re}^{-1/2}\frac{\partial^2 u_{1a}}{\partial Y_a^2},\quad \frac{\partial p_{1a}}{\partial Y_a}=0,\quad \frac{\partial u_{1a}}{\partial X}+\frac{\partial v_{1a}}{\partial Y_a}=0 \tag{2.8}$$

for the functions of the first approximation.

In the first equation of (2.8), the viscous and inertial terms are of the same order if the small parameter α (the amplitude of the perturbations of the longitudinal velocity) and the Reynolds number Re are connected by relation (1.9). This case has been analysed previously [6–10]. In the following analysis, we shall dwell on the case when

$$\mathrm{Re}^{-1/7} \ll \alpha \ll 1 \tag{2.9}$$

Conditions (2.9) imply a more complex flow structure compared with that known from [6–10]. First, it should be noted that the coefficient $\alpha^{-7/2}\mathrm{Re}^{-1/2}$ of the highest derivative in the first equation of (2.8) becomes a small parameter. Consequently, under assumption (2.9), domain *3* is described by the system of equations

$$\frac{\partial u_{1a}}{\partial T} + u_{1a}\frac{\partial u_{1a}}{\partial X} + v_{1a}\frac{\partial u_{1a}}{\partial Y_a} = -\frac{\partial p_{1a}}{\partial X}, \quad \frac{\partial p_{1a}}{\partial Y_a} = 0, \quad \frac{\partial u_{1a}}{\partial X} + \frac{\partial v_{1a}}{\partial Y_a} = 0 \tag{2.10}$$

The conditions for matching with solution (2.2) in the lower boundary of domain *2* ($Y_m \to 0$), that is, the limiting expressions (2.5), serve as the asymptotic boundary conditions for Eqs (2.10) in the upper boundary of domain *3* ($Y_a \to +\infty$). Rewriting expressions (2.5) in the variable Y_a (2.6), we obtain the required asymptotic form of the solutions of the system of equations (2.10)

$$Y_a \to +\infty: u_{1a} \to \lambda_1(Y_a + A_1), \quad v_{1a} \to -\lambda_1\frac{\partial A_1}{\partial X}Y_a \tag{2.11}$$

and matching with expansions (2.5) gives

$$p_{1a} = -\Delta\frac{\partial^2 A_1}{\partial X^2}, \quad \Delta = \int_0^{+\infty} U_0^2(\xi)d\xi \tag{2.12}$$

for the pressure.

It can be shown that the functions

$$u_{1a} = \lambda_1(Y_a + A_1), \quad v_{1a} = -\lambda_1\frac{\partial A_1}{\partial X}Y_a - \frac{\partial A_1}{\partial T} - \lambda_1 A_1\frac{\partial A_1}{\partial X} - \frac{1}{\lambda_1}\frac{\partial p_{1a}}{\partial X} \tag{2.13}$$

satisfy both Eqs (2.10) and boundary conditions (2.11).

The perturbations in domain *2* and, when condition (2.9) is satisfied, in domain *3* are therefore described by the inviscid equations (2.3) and (2.10); the solutions of these equations are expressed by formulae (2.4) and (2.13) which contain an arbitrary function $A_1(T, X)$. However, the first equality of (2.13) shows that one of the non-slip conditions, in fact, $u_{1a} = 0$, is not satisfied on the wall $Y_a = 0$. This is evidence of the existence of a viscous sublayer (domain *4* in Fig. 1), immediately adjacent to the wall, the thickness of which is much smaller than the thickness of domain *3*.

The term with the second derivative on the right-hand side of the first equation of system (2.8) becomes of the order of unity at distances $Y_a = O(\alpha^{-7/4}\mathrm{Re}^{-1/4})$ from the solid wall. The above estimate, by determining the thickness of the viscous sublayer, dictates the need to introduce a new vertical coordinate

$$Y_l = \alpha^{7/4}\mathrm{Re}^{1/4}Y_a \tag{2.14}$$

The viscous domain *4* is characterized by the condition $Y_l = O(1)$, and the flow parameters in this domain are represented in the form

$$\tilde{u} = \alpha u_{1l}(T, X, Y_l) + \dots, \quad \tilde{v} = \alpha^{3/4}\mathrm{Re}^{-1/4}v_{1l}(T, X, Y_l) + \dots, \quad \tilde{p} = \alpha^2 p_{1l}(T, X, Y_l) + \dots \tag{2.15}$$

Introducing expressions (2.15) into the Navier–Stokes equations (1.2) and retaining terms of the basic order of magnitude, we arrive at the Prandtl equations

$$\frac{\partial u_{1l}}{\partial T} + u_{1l}\frac{\partial u_{1l}}{\partial X} + v_{1l}\frac{\partial u_{1l}}{\partial Y_l} = -\frac{\partial p_{1l}}{\partial X} + \frac{\partial^2 u_{1l}}{\partial Y_l^2}, \quad \frac{\partial p_{1l}}{\partial Y_l} = 0, \quad \frac{\partial u_{1l}}{\partial X} + \frac{\partial v_{1l}}{\partial Y_l} = 0 \tag{2.16}$$

The solid surface, on which the following boundary condition are obvious

$$Y_l = 0: u_{1l} = \upsilon_{1l} = 0 \tag{2.17}$$

serves as the lower boundary of domain *4*.

The limiting condition on approaching the upper boundary $Y_l \to +\infty$ of domain *4* from inside follows as a result of matching with solution (2.13) on approaching this boundary from outside of domain *3*, which corresponds in view of relations (2.9) and (2.14). Letting $Y_a \to 0$ in equalities (2.13), we obtain

$$Y_l \to +\infty: u_{1l} \to \lambda_1 A_1(T, X) \tag{2.18}$$

The second equation of system (2.16) enables us to match the pressure: $p_{1l}(T,X) = p_{1a}(T,X)$. Taking expression (2.12) into account is necessary to assume

$$p_{1l}(T, X) = -\Delta \frac{\partial^2 A_1(T, X)}{\partial X^2} \tag{2.19}$$

We will now show that the problem of constructing the flow field in domains *2* and *3* can be solved independently of the problem of the integration of problem (2.16) in domain *4*. For this purpose, we return to the matching of the velocity components υ_{1a} and υ_{1l}. It can be seen from the representations (2.15) that when $Y_a \to 0$ the function $\tilde{\upsilon}$ must be a quantity $O(\alpha^{3/4}\mathrm{Re}^{-1/4})$ and comparison with (2.7) gives

$$Y_a \to 0: \upsilon_{1a} = O(\alpha^{-7/4}\mathrm{Re}^{-1/4}) \tag{2.20}$$

By virtue of inequality (2.9), the matching condition (2.20) is equivalent, apart from a small parameter $\alpha^{-7/4}\mathrm{Re}^{-1/4}$, to the impermeability condition: $\upsilon_{1a} = 0$ when $Y_a = 0$. However, this last condition can only be satisfied when the function $A_1(T, X)$ satisfies the Korteweg–de Vries equation

$$\frac{\partial A_1}{\partial T} + \lambda_1 A_1 \frac{\partial A_1}{\partial X} = \frac{\Delta}{\lambda_1}\frac{\partial^3 A_1}{\partial X^3} \tag{2.21}$$

This assertion follows from expressions (2.13) for υ_{1a} and (2.12) for p_{1a}. If the function $A_1(T, X)$ is found from Eq. (2.21), then the field of the perturbed flow in domains *2* and *3* is constructed using formulae (2.4) and (2.12), (2.13) respectively.

Hence, the viscous boundary domain *4* plays a passive role in the formation of the inviscid flow in domains *2* and *3*. The velocity field in domain *4* is determined from the solution of the classical problem (2.16)–(2.18) for a system of Prandtl equations with a specified pressure gradient (since the function $A_1(T, X)$ in relations (2.18) and (2.19) is known).

3. NON-LINEAR PERTURBATIONS IN THE FORM OF CNOIDAL WAVES

We will assume that condition (2.9), subject to which a quadruple-level structure of the perturbed velocity field is realized, is satisfied and that the function for the displacement of the streamlines $A_1(T, X)$ from relations (2.4), (2.12) and (2.13) satisfies the Korteweg–de Vries equation (2.21).

We eliminate the constants Δ and λ_1 from the following equations by changing to the new variables

$$\begin{gathered} t = \Delta^{-2/7}\lambda_1^{8/7}T, \quad x = \Delta^{-3/7}\lambda_1^{5/7}X, \quad y_a = \Delta^{-1/7}\lambda_1^{4/7}Y_a \\ u = \Delta^{-1/7}\lambda_1^{-3/7}u_{1a}, \quad \upsilon = \Delta^{1/7}\lambda_1^{-4/7}\upsilon_{1a}, \quad p = \Delta^{-2/7}\lambda_1^{-6/7}p_{1a}, \quad A = \Delta^{-1/7}\lambda_1^{4/7}A_1 \end{gathered} \tag{3.1}$$

In variables (3.1), we have

$$\frac{\partial A}{\partial t} + A\frac{\partial A}{\partial x} = \frac{\partial^3 A}{\partial x^3} \tag{3.2}$$

Assuming

$$A(x,t) = \Phi_0(\xi_0), \quad \xi_0 = x - c_0 t \tag{3.3}$$

in the Korteweg–de Vries equation (3.2), after a single integration (the constant of integration is taken to be equal to zero, which enables us to look for solutions which are decaying at infinity: $\Phi_0 \to 0$, $|\xi_0| \to \infty$) we obtain

$$\frac{d^2\Phi_0}{d\xi_0^2} = -c_0\Phi_0 + \frac{\Phi_0^2}{2} \tag{3.4}$$

Equation (3.4) is invariant under the single parameter group of transformations

$$\Phi_0 \to \gamma_0\Phi_0, \quad \xi_0 \to \gamma_0^{-1/2}\xi_0, \quad c_0 \to \gamma_0 c_0 \tag{3.5}$$

Hence, the choice of the parameter $\gamma_0 = |c_0|$ in (3.5) enables us to eliminate c_0 from Eq. (3.4) by means of the substitution

$$\Phi_0 = |c_0|\Phi(\xi), \quad \xi = |c_0|^{1/2}\xi_0 \tag{3.6}$$

where, for the unknown function $\Phi(\xi)$, we have the equation

$$\frac{d^2\Phi}{d\xi^2} = -\Phi\,\mathrm{sign}\,c_0 + \frac{\Phi^2}{2} \tag{3.7}$$

We shall subsequently assume that $c_0 < 0$ since a change in the sign in front of the linear term on the right-hand side of Eq. (3.7) is achieved by the substitution $\Phi = \tilde{\Phi} + 2\,\mathrm{sign}\,c_0$. As is customary, on introducing

$$\Theta = \frac{d\Phi}{d\xi}, \quad \frac{d^2\Phi}{d\xi^2} = \Theta\frac{d\Theta}{d\Phi}$$

we reduce the order of Eqs (3.7):

$$\Phi^2 + \frac{\Phi^3}{3} = \Theta^2 + K \tag{3.8}$$

The choice of the constant $K = 0$ in Eq. (3.8) corresponds to the separatrix in the phase plane $\{\Phi, \Theta\}$ and leads to the equation

$$\frac{d\Phi}{d\xi} = \pm\Phi\sqrt{1 + \frac{\Phi}{3}} \tag{3.9}$$

We will consider the case when $\Phi < 0$; then integration of Eq. (3.9) gives the soliton solution (tending exponentially to zero when $\xi \to \pm\infty$)

$$\Phi = -3\,\mathrm{ch}^{-2}\frac{\xi - \xi_{00}}{2} \tag{3.10}$$

Taking account of relations (3.3) and (3.6), we conclude that the soliton of the Korteweg–de Vries equation is determined by the following expression, which includes the two parameters c_0 and x_0,

$$A = -3|c_0|\,\mathrm{ch}^{-2}\left[\frac{|c_0|^{1/2}}{2}(x + |c_0|t - x_0)\right] \tag{3.11}$$

So, the soliton of the Korteweg–de Vries equation moves to the left without changing its form and has an amplitude which is three times greater than the modulus of its phase velocity.

Suppose β_1, β_2 and β_3 are roots of the polynomial

$$\Phi^3 + 3\Phi^2 - 3K = (\Phi - \beta_1)(\Phi - \beta_2)(\Phi - \beta_3) \tag{3.12}$$

Soliton (3.10) considered above is represented by a phase trajectory of Eq. (3.8) in the form of a loop containing the two separatrices of the saddle point $\Phi = \Theta = 0$ and which corresponds to the multiple root of polynomial (3.12) for $K = 0$: $\beta_1 = \beta_2 = 0$, $\beta_3 = -3$ where $\beta_1 > \Phi > \beta_3$. Closed trajectories, corresponding to $\beta_1 > 0 > \beta_2 > \beta_3$ for $4/3 > K > 0$, envelop the stationary point for $K = 4/3$ of the centre type $\Phi = -2$, $\Theta = 0$. In this case, the periodic solutions of Eq. (3.7) are described by an elliptic integral and, after introducing the notation

$$\sin^2\chi = \frac{\beta_2 - \beta_3}{\beta_1 - \beta_3}, \quad \sin^2\varphi = \frac{\Phi - \beta_3}{\beta_2 - \beta_3} \tag{3.13}$$

and replacing the variable

$$\Phi = \beta_3 + (\beta_2 - \beta_3)\sin^2\theta$$

we have

$$\xi - \xi_{00} = \frac{2\sqrt{3}}{\sqrt{\beta_1 - \beta_3}}\int_0^{\varphi}\frac{d\theta}{\sqrt{1 - \sin^2\chi\sin^2\theta}} \tag{3.14}$$

Suppose the constant K in Eq. (3.8) is such that

$$K = 4/3 - \tilde{\varepsilon}, \quad \tilde{\varepsilon} \to +0 \tag{3.15}$$

It is convenient to seek the roots of polynomial (3.12) by rewriting it in terms of the new variable: $\Phi = -2 + \tilde{\Phi}$. Expression (3.12) vanishes when

$$\tilde{\Phi}^3 - 3\tilde{\Phi}^2 + 3\tilde{\varepsilon} = 0 \tag{3.16}$$

It can be seen that, for small $\tilde{\varepsilon}$, the largest root of Eq. (3.16) is located close to the point $\tilde{\Phi} = 3$ and the two other roots lie in the left-hand and right-hand neighbourhoods of the point $\tilde{\Phi} = 0$. More precisely, under assumption (3.15), we obtain from equality (3.16) that

$$\beta_1 = 1 + \frac{1}{3}\tilde{\varepsilon} + O(\tilde{\varepsilon}^2), \quad \beta_2 = -2 + \sqrt{\tilde{\varepsilon}} + \frac{1}{6}\tilde{\varepsilon} + O(\tilde{\varepsilon}^{3/2}), \quad \beta_3 = -2 - \sqrt{\tilde{\varepsilon}} + \frac{1}{6}\tilde{\varepsilon} + O(\tilde{\varepsilon}^{3/2}), \quad \tilde{\varepsilon} \to +0 \tag{3.17}$$

We now substitute the asymptotic form (3.17) into Eqs (3.13) and (3.14) which describe a closed phase trajectory. Since, from the first relation of (3.13), we have

$$\sin^2\chi = \frac{2}{3}\sqrt{\tilde{\varepsilon}} + O(\tilde{\varepsilon}) \tag{3.18}$$

equality (3.14) can be rewritten in the form

$$\xi - \xi_{00} = \pm 2\left[\varphi - \frac{1}{12}\sqrt{\tilde{\varepsilon}}\sin 2\varphi + O(\tilde{\varepsilon})\right] \tag{3.19}$$

The second relation of (3.13) gives

$$\Phi = -2 - \sqrt{\tilde{\varepsilon}} + 2\sqrt{\tilde{\varepsilon}}\sin^2\varphi + O(\tilde{\varepsilon}) \tag{3.20}$$

For $\tilde{\varepsilon} \to +0$, the relation $\xi = \xi(\varphi)$, which is defined by formula (3.19), can be inverted

$$\varphi = \pm\left[\frac{1}{2}(\xi - \xi_{00}) + \frac{1}{12}\sqrt{\tilde{\varepsilon}}\sin(\xi - \xi_{00}) + O(\tilde{\varepsilon})\right] \tag{3.21}$$

It remains to introduce expression (3.21) into the right-hand side of Eq. (3.20) in order to obtain an explicit form of solutions (3.14), which are described by closed phase trajectories in a small neighbourhood of the singular point $\Phi = -2$, $\Theta = 0$ of the centre type:

$$\Phi = -2 + \frac{1}{4}\tilde{\varepsilon} - \sqrt{\tilde{\varepsilon}}\cos(\xi - \xi_{00}) - \frac{1}{12}\tilde{\varepsilon}\cos 2(\xi - \xi_{00}) + O(\tilde{\varepsilon}^{3/2}) \tag{3.22}$$

We obtain the three-parameter family of solutions of the Korteweg–de Vries equation from relation (3.22) having made use of relations (3.3) and (3.6) and the notation (k is the wave number)

$$k = |c_0|^{1/2}, \quad kx_0 = \xi_{00} + \pi, \quad \hbar = k^2\sqrt{\tilde{\varepsilon}} \to 0$$

A periodic travelling wave with a negative phase velocity is defined by the expression

$$A(t,x) = -2k^2 + \frac{\hbar^2}{4k^2} + \hbar\cos[k(x - x_0 + k^2 t)] - \frac{\hbar^2}{12k^2}\cos[2k(x - x_0 + k^2 t)] + O(\hbar^3) \tag{3.23}$$

The Korteweg–de Vries equation is invariant under Galilean transformations (changing to a system of coordinates moving with a velocity D_0 in the negative direction of the x axis)

$$x \to x - D_0 t, \quad A \to D_0 + A \tag{3.24}$$

The group property (3.24) of the Korteweg–de Vries equation enable us to introduce a further parameter into expression (3.23) and to drop the assumption of negative phase velocities. Putting $D_0 = k^2 + c$ in relations (3.24), we obtain a four-parameter family of wave solutions of the Korteweg–de Vries equation with phase velocities c of any sign

$$A(t,x) = c - k^2 + \frac{\hbar^2}{4k^2} + \hbar\cos[k(x - x_0 - ct)] - \frac{\hbar^2}{12k^2}\cos[2k(x - x_0 - ct)] + O(\hbar^3) \tag{3.25}$$

If $A(t,x)$ is the solution of the Korteweg–de Vries equation (3.2), then the substitution

$$A \to \vartheta A, \quad t \to \vartheta^{-3/2} t, \quad x \to \vartheta^{-1/2} x \tag{3.26}$$

leads to the solution of the same equation. Hence, all the solutions (3.23) and (3.25) are obtained from the steady solution of the Korteweg–de Vries equation

$$A = -1 + \frac{\hbar^2}{4} + \hbar\cos x - \frac{\hbar^2}{12}\cos 2x + O(\hbar^3) \tag{3.27}$$

by applying the group transformations (3.24) and (3.26).

The function $\mathrm{am}(\Omega\,|\,\mathbb{N})$, which is the inverse of the function

$$\Omega(\varphi\,|\,\mathbb{N}) = \int_0^{\varphi} \frac{d\theta}{\sqrt{1 - \mathbb{N}\sin^2\theta}}$$

is called a Jacobi amplitude [11] and the function

$$\mathrm{cn}(\Omega\,|\,\mathbb{N}) = \cos[\mathrm{am}(\Omega\,|\,\mathbb{N})]$$

is known as a Jacobi elliptic cosine (a cnoid). The above-mentioned functions enable us to represent the implicit relation (3.13), (3.14) in the form

$$\Phi = \beta_2 - (\beta_2 - \beta_3)\mathrm{cn}^2\left\{\left[\sqrt{\frac{\beta_1 - \beta_3}{12}}(\xi - \xi_{00})\right]\middle|\,\mathbb{N}\right\} \tag{3.28}$$

The quarter period $K = K(\mathbb{N})$ of the function $\mathrm{cn}(\Omega\,|\,\mathbb{N})$ and the parameter $\mathbb{N}$ are determined by means of (3.13) and (3.14), namely

$$K(\mathbb{N}) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\mathbb{N}\sin^2\theta}}, \quad \mathbb{N} = \sin^2\chi = \frac{\beta_2-\beta_3}{\beta_1-\beta_3}$$

Periodic non-linear solutions of the type (3.28), which are called cnoidal waves [12, 13], possess, as has been shown above, the representation (3.25) in the weakly non-linear limit of small amplitudes $\beta_2 - \beta_3 = O(\sqrt{\tilde{\varepsilon}})$, $\tilde{\varepsilon} \to +0$.

4. NARROWING THE CLASS OF POSSIBLE SOLUTIONS BY TAKING ACCOUNT OF THE VISCOUS BOUNDARY

The class of oscillatory solutions of the non-linear system of equations (2.10) which is defined by Eq. (3.8) when $0 < K < 4/3$ and, in the case of small amplitudes $\hbar$, admits of representations (3.25) and (3.27), describes the motion in the domain $Y_a = O(1)$. However, solutions from this class do not satisfy the no-slip condition $u_a = 0$ on the wall $Y_a = 0$, as can be seen from the first equality of (2.13). This fact is indicative of the existence, which has already been noted above, of a thin sublayer $Y_l = O(1)$ in the flow with an estimate of its thickness (2.14), in which the structure of the velocity field is formed by viscous shear stresses (domain *4* in Fig. 1).

We will also rewrite boundary condition (2.16)–(2.19) in the new variables (3.1) having added the quantities

$$u_l = \Delta^{-1/7}\lambda_1^{-3/7}u_{1l}, \quad v_l = \Delta^{1/7}\lambda_1^{-4/7}v_{1l}, \quad p_l = \Delta^{-2/7}\lambda_1^{-6/7}p_{1l}, \quad y = \Delta^{-1/7}\lambda_1^{4/7}Y_l$$

to them.

Then, for the viscous boundary sublayer, we have

$$\frac{\partial u_l}{\partial t} + u_l\frac{\partial u_l}{\partial x} + v_l\frac{\partial u_l}{\partial y} = -\frac{\partial p_l}{\partial x} + \frac{\partial^2 u_l}{\partial y^2}, \quad \frac{\partial p_l}{\partial y} = 0, \quad \frac{\partial u_l}{\partial x} + \frac{\partial v_l}{\partial y} = 0$$
$$y = 0\colon u_l = v_l = 0; \quad y \to +\infty\colon u_l \to A(t, x) \tag{4.1}$$

As a consequence of Eq. (2.19), the pressure in relations (4.1) is the specified function

$$p_l(t, x) = -\frac{\partial^2 A(t, x)}{\partial x^2} \tag{4.2}$$

We take the solution of the Korteweg–de Vries equation (3.25) as the outer boundary condition for the inner problem (4.1), (4.2) in the viscous boundary sublayer. It will be shown below that a periodic solution in the viscous sublayer does not exist for all c and k in Eq. (3.25) but only in the case when a definite relation between them in satisfied. Since Eqs (4.1) are also invariant under the transformations (3.24):

$$x \to x - D_0 t, \quad A \to D_0 + A, \quad u_l \to D_0 + u_l \tag{4.3}$$

we will change to a system of coordinates where the wave (3.25) is stationary, that is, in relations (3.24) and (3.25), we put $D_0 = -c$. We now also turn our attention to the invariance of problem (4.1) with respect to the one-parameter group of transformations

$$t \to \vartheta^{-3/2}t, \quad x \to \vartheta^{-1/2}x, \quad y \to \vartheta^{-3/4}y, \quad u_l \to \vartheta u_l, \quad v_l \to \vartheta^{3/4}v_l, \quad p_l \to \vartheta^2 p_l \tag{4.4}$$

The application of these transformations implies that the boundary function $A(t, x)$, occurring in relations (4.1) and (4.2), transforms in accordance with the substitution (3.26).

In Eq. (3.25), we now discard the unimportant constant x_0 and make the replacement (3.24), by putting $D_0 = -c$ and, also, the substitution (3.26), by choosing $\vartheta = k^{-2}$. As a result, by redesignating $\hbar \to \hbar k^2$, we obtain Eq. (3.27). We now sequentially carry out the transformations (4.3) and (4.4) as applied to problem (4.1) with exactly the same choice of D_0 and ϑ. The construction of the inner solution in the viscous boundary sublayer, which is generated by the outer boundary-value condition (3.25), reduces to solving the steady boundary-value problem

$$u_l\frac{\partial u_l}{\partial x}+v_l\frac{\partial u_l}{\partial y} = -\frac{\partial p_l}{\partial x}+\frac{\partial^2 u_l}{\partial y^2},\quad \frac{\partial p_l}{\partial y} = 0,\quad \frac{\partial u_l}{\partial x}+\frac{\partial v_l}{\partial y} = 0$$
$$y = 0{:}\ u_l = c_w,\quad v_l = 0;\quad y\to+\infty{:}\ u_l\to A(x) \tag{4.5}$$

The parameter $c_w = -ck^{-2}$ is the velocity of the wall along the tangent to itself, the function $A(x)$ is given by expression (3.27) and the pressure gradient is calculated using the steady-state analogue of relation (4.2). It will be subsequently seen that the quantity c_w cannot be arbitrarily specified.

So, in the case when

$$A = -1+\frac{\hbar^2}{4}+\frac{\hbar}{2}e^{ix}+\frac{\hbar}{2}e^{-ix}-\frac{\hbar^2}{24}e^{2ix}-\frac{\hbar^2}{24}e^{-2ix}+\ldots$$
$$\frac{\partial p_l}{\partial x} = \frac{i\hbar}{2}e^{ix}-\frac{i\hbar}{2}e^{-ix}-\frac{i\hbar^2}{3}e^{2ix}+\frac{i\hbar^2}{3}e^{-2ix}+\ldots \tag{4.6}$$

we introduce the constants m_0, m_1 and m_2, which are to be determined and consider the expansion

$$c_w = m_0+\hbar m_1+\hbar^2 m_2+O(\hbar^3) \tag{4.7}$$

It is clear that, for $m_0\neq -1$, the solution of problem (4.5), (4.6) in the case when $\hbar = 0$ is expressed in terms of the Blasius function, which depends on the self-similar variable $yx^{-1/2}$ and is not periodic. It therefore follows that $m_0 = -1$ is taken. The last equality enables us to seek the solution of problem (4.5), (4.6) in the form

$$u_l = -1+\hbar q_0'(y)+\hbar^2 f_0'(y)-\hbar f_1'(y)e^{ix}-\hbar\bar{f}_1' e^{-ix}-\hbar^2 f_2'(y)e^{2ix}-\hbar^2\bar{f}_2'(y)e^{-2ix}+O(\hbar^3)$$
$$v_l = i\hbar f_1(y)e^{ix}-i\hbar\bar{f}_1(y)e^{-ix}+2i\hbar^2 f_2(y)e^{2ix}-2i\hbar^2\bar{f}_2(y)e^{-2ix}+O(\hbar^3) \tag{4.8}$$

where a bar above a symbol denotes a complex conjugate and a prime denotes a derivative with respect to y. The equations

$$f_1'''+if_1' = -i/2,\quad q_0''' = 0 \tag{4.9}$$

are true in the first approximation in the parameter $\hbar$.

The boundary conditions for Eqs (4.9) follow from relations (4.5), (4.6) and (4.8)

$$y\to+\infty{:}\ f_1'\to -1/2,\quad q_0'\to 0 \tag{4.10}$$

$$y = 0{:}\ f_1 = f_1' = 0,\quad q_0' = m_1 \tag{4.11}$$

From the second equation of (4.9) and the second limiting condition of (41.10), we obtain $q_0'\equiv 0$. This means that $m_1 = 0$. The first equation of (4.9) gives

$$f_1' = -1/2+M_1^+e^{\tau y}+M_1^-e^{-\tau y},\quad \tau = (1-i)/\sqrt{2} \tag{4.12}$$

The requirement that there is no exponential growth when $y\to+\infty$ on the right-hand side of Eq. (4.12) is sufficient to satisfy the limiting condition (4.10) and, therefore, the constant $M_1^+ = 0$, and the other constant in Eq. (4.12) is determined by the first boundary condition of (4.11), that is, $M_1^- = 1/2$. Integrating Eq. (4.12), taking account of the first boundary condition of (4.11), we obtain

$$f_1 = -y/2+(1-e^{-\tau y})/(2\tau) \tag{4.13}$$

In the second approximation in the parameter $\hbar$, substitution of expressions (4.8) into Eqs (4.15) leads to the equations

$$f_2''' + 2if_2' = i/3 - if_1'^2 + if_1f_1', \quad f_0''' = -if_1\bar{f}_1'' + i\bar{f}_1f_1'' \tag{4.14}$$

As the boundary conditions for Eqs (4.14), from relations (4.6) and (4.8) we have

$$y \to +\infty: f_2' \to 1/24, \quad f_0' = 1/4 \tag{4.15}$$

$$y = 0: f_2 = f_2' = 0, \quad f_0' = m_2 \tag{4.16}$$

We shall initially consider only the first equation of (4.14) which, after substitution of the function f_1 of the first approximation of (4.13) into its right-hand side, has a general solution of the form

$$f_2' = 1/24 + (\tau y - 1)e^{-\tau y}/4 + M_2^+ e^{\tau\sqrt{2}y} + M_2^- e^{-\tau\sqrt{2}y}$$

(M_2^+ and M_2^- are arbitrary constants). We now eliminate the exponentially increasing term from this solution by putting $M_2^+ = 0$, after which we will show that the first limiting condition of (4.15) is satisfied. The equality $M_2^- = 5/24$ ensures that the first boundary condition of (4.16) is satisfied. Then, by again taking account of the first condition of (4.16), we obtain

$$f_2 = [y(1 - 6e^{-\tau y}) + 5(1 - e^{-\tau\sqrt{2}y})/(\tau\sqrt{2})]/24 \tag{4.17}$$

We now consider the second equation of (4.14). The first condition of (4.11) enables us to simplify this equation by integrating its right-hand side by parts. We have

$$f_0''(y) - f_0''(0) = i\bar{f}_1(y)f_1'(y) - if_1(y)\bar{f}_1'(y) \tag{4.18}$$

Using the notation

$$I(y) \equiv \int_0^y [i\bar{f}_1(s)f_1'(s) - if_1(s)\bar{f}_1'(s)]ds$$

from expression (4.13) we derive the equality

$$I(y) = \frac{1}{4}[3 - y\sqrt{2} + e^{-\sqrt{2}y} - (\tau y + 2 + i)e^{-\tau y} - (\bar{\tau}y + 2 - i)e^{-\bar{\tau}y}] \tag{4.19}$$

The equivalent way of writing Eq. (4.18) has the form

$$f_0'(y) = E_0 + E_1 y + I(y); \quad E_0 = f_0'(0), \quad E_1 = f_0''(0) \tag{4.20}$$

where E_0 and E_1 are arbitrary constants. We choose $E_1 = \sqrt{2}/4$. Then, in accordance with expression (4.19), there will be no linearly increasing terms on the right-hand side of Eq. (4.23), that is, a finite limit

$$\lim_{y \to +\infty} [E_1 y + I(y)] = 3/4 \tag{4.21}$$

exists.

On the other hand, the asymptotic condition (4.15) shows that the limit of (4.21), which is being considered, is equal to

$$\lim_{y \to +\infty} [f_0'(y) - f_0'(0)] = 1/4 - E_0 \tag{4.22}$$

Comparison of relations (4.21) and (4.22) gives $E_0 = -1/2$. Hence, the constants E_0, and E_1, and thereby also the function $g_0 = f_0'(y)$, are found in a unique manner from the second equation of (4.14) and the second limiting condition of (4.15). Since the boundary condition (4.16) does not participate in the definition of the function g_0, it can only be satisfied in the case of a special selection of the parameter m_2.

So, by putting $y = 0$ in Eq. (4.20), in accordance with condition (4.16), we find

$$m_2 = -1/2 \tag{4.23}$$

Consequently, a solution of problem (4.5), (4.6) which is 2π-periodic in the variables x exists in the domain $y > 0$ and is expressed by formulae (4.8), (4.13), (4.17), (4.19) and (4.20) if the tangential velocity c_w of the boundary of the domain $y = 0$, appearing in problem (4.5), is related to the amplitude $\hbar$ of the limiting functions from the expressions (4.6) as follows:

$$c_w = -1 - \hbar^2/2 + O(\hbar^3) \tag{4.24}$$

We will now change to a system of coordinates in which the wall is stationary, making use of the fact that the system of Prandtl equations in invariant under the transformations (4.3). To do this, it is sufficient to put $D_0 = |c_w|$ in relation (4.3), where c_w is given by expression (4.24). As a result, we obtain a non-stationary solution, which is periodic in both x and t. Finally, the period of the oscillatory solution can be chosen arbitrarily due to the invariance of the Prandtl equations under the transformations (4.4). In relation (4.4), we now put $\hat{\hbar} = k^2\hbar$ and use the notation $\vartheta = k^2$. Successive application of the transformations (4.3) and (4.4), with the parameters D_0 and ϑ mentioned above, generates a family of solutions of problem (4.1) which depends on the three parameters c, k and $\hat{\hbar}$. When $y_l \to +\infty$, this family of solutions possesses the asymptotic form $u_l \to A(x, t)$, where the limiting function $A(x, t)$ is obtained from the first expression of (4.6) by means of the above-mentioned transformations (4.3) and (4.4)

$$A(t, x) = \frac{3\hat{\hbar}^2}{4k^2} + \hat{\hbar}\cos[k(x-ct)] - \frac{\hat{\hbar}^2}{12k^2}\cos[2k(x-ct)] + O(\hat{\hbar}^3) \tag{4.25}$$

and, moreover, as a consequence of Eq. (4.24), the parameters c, k and $\hat{\hbar}$ are not independent but are subject to the relation

$$c = k^2 + \frac{\hat{\hbar}^2}{2k^2} + O(\hat{\hbar}^3) \tag{4.26}$$

The function (4.25) is the solution of the Korteweg–de Vries equation (since this equation is invariant under the transformations (3.24) and (3.26)). Expression (4.25) is a contraction of the three-parameter family of solutions (3.25) due to relation (4.26). This relation arises from the condition for a periodic solution to exist in the viscous boundary sublayer *4* (the inner solution for which (3.25) is the outer solution relating to domain *3*).

We now point out another method for determining the constant (4.23) starting from the necessary condition for the existence of a 2π-periodic solution of the Prandtl boundary layer equations, formulated by Wood [14]. For this purpose, we note that, by means of the transformation

$$x = -\hat{x}, \quad y = |c_w|^{-1/2}\hat{y}, \quad u = -|c_w|\hat{u}, \quad v = |c_w|^{1/2}\hat{v}, \quad p = |c_w|^2\hat{p}, \quad A = -|c_w|U_\infty$$

problem (4.5) can be reduced to the problem of a classical boundary layer. The quantity c_w in chosen in accordance with expressions (4.7) and, as was shown above, $m_0 = -1$, $m_1 = 0$.

Actually, in the case of the problem of a classical boundary layer, the above-mentioned condition for a 2π-periodic solution to exist [14] states that

$$\int_0^{2\pi} (U_\infty^3 - U_\theta)d\hat{x} = O(\hbar^4) \tag{4.27}$$

where $U_\infty(\hat{x})$ is a specified (2π-periodic) function.

We now substitute the expansion of the function A, in accordance with the first expression of (4.6) and the representation (4.7) of the parameter c_w, into the limiting function $U_\infty = -|c_w|A^{-1}$. We then obtain

$$U_\infty^3 - U_\infty = 2m_2\hbar^2 - 2\hbar\cos x + 2\hbar^2\cos^2 x + \frac{2\hbar^2}{3}\cos 2x + O(\hbar^3) \tag{4.28}$$

as the integrand in Eq. (4.27).

The integral along the segment $[0, 2\pi]$ of terms up to $O(\hbar^2)$ inclusive occurring in the function (4.28) has to be equated to zero in accordance with condition (4.27), which leads to a value for the unknown constant m^2 which is identical with the value (4.23).

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